SPICE-Based Statistical Assessment of Interconnects Terminated by Nonlinear Loads with Polynomial Characteristics

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***Abstract*—This paper proposes an exact formalism for the inclusion of static nonlinear elements with polynomial I-V characteristic into the polynomial chaos framework for statistical circuit simulation, which was so far limited to linear circuits. The formulation is SPICE-compatible, thus allowing the convenient integration of such nonlinear elements into standard circuit solvers. This contribution represents a step forward towards the inclusion of nonlinear terminations into the SPICE- and polynomial chaos-based statistical analysis of interconnects with stochastic parameters. The theory is illustrated and validated by means of an application example.**

***Index Terms*—Circuit modeling, circuit simulation, nonlinear, polynomial chaos, stochastic analysis, transmission lines, uncertainty.**

I. Introduction

With the increasing shrinking of device dimensions, the impact of process variability on circuit performance is becoming more and more important. Therefore, statistical approaches are usually preferred in circuit simulation in order to provide variation-aware results and thus more robust designs [1], [2]. In the packaging and manufacturing community, great attention has been attracted so far by polynomial chaos (PC)-based techniques [3]–[8], according to which statistical information is obtained via the projection of stochastic variables onto orthogonal polynomials [9].

Specifically, the authors of this contribution developed a PC-based framework for the statistical simulation of distributed networks that include lossy and dispersive lines with variability in their cross-sectional parameters [10]. The formulation is compatible with standard SPICE-type simulators, thus easily enabling the simulation of arbitrary network topologies. Nevertheless, a considerable limitation of the approach is that it hitherto applies exclusively to linear circuits.

As far as the extension towards nonlinear networks is concerned, a novel formalism has been proposed allowing for the inclusion of general nonlinear terminations into the PC framework [11]. Such a new formulation has been implemented into MATLAB in conjunction with a finite-difference time-domain (FDTD) scheme and applies to arbitrary I-V characteristics. However, although very good accuracy was established, the approach is approximate. In this paper, we show that an alternative and *exact* formulation can be derived when the nonlinear terminations have a static polynomial I-V characteristic. We also demonstrate that its integration in SPICE solvers is straightforward, thus allowing the analysis of arbitrary network topologies as well as the inclusion of lossy and dispersive interconnects, which would be rather cumbersome via the FDTD technique.

The paper is organized as follows: Section II summarizes the rationale of the PC-based circuit simulation; Section III presents the formalism for nonlinear elements with polynomial I-V characteristic and discusses the integration into SPICE solvers; numerical results and validations are provided in Section IV; finally, conclusions are drawn in Section V.

II. Polynomial Chaos-Based Simulation of Stochastic Interconnects

For the sake of simplicity and ease of notation, the discussion is based on a single transmission line characterized by stochastic per-unit-length (p.u.l.) parameters $R(\boldsymbol{\xi})$, $L(\boldsymbol{\xi})$, $G(\boldsymbol{\xi})$, $C(\boldsymbol{\xi})$, and loaded with a given termination, as illustrated in Fig. 1. Here, $\boldsymbol{\xi} \in \mathbb{R}^d$ is a d-variate vector collecting all the independent random variables (RVs) affecting the line properties and is used to highlight the parameters that exhibit variability. Generalization to multiconductor interconnects, as well as to larger network topologies, will be straightforward.

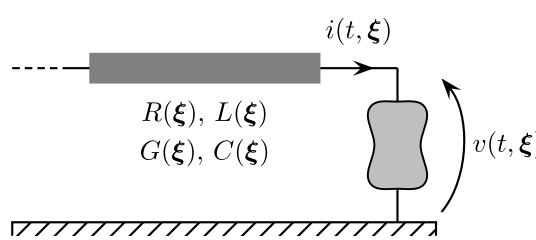

Fig. 1. Stochastic transmission line and its termination.

The underlying idea of the PC-based simulation of stochastic interconnects is to express terminal voltages and currents as PC expansions:

$$v(t,\boldsymbol{\xi}) \approx \sum_{k=0}^{P} v_k(t)\phi_k(\boldsymbol{\xi}), \quad i(t,\boldsymbol{\xi}) \approx \sum_{k=0}^{P} i_k(t)\phi_k(\boldsymbol{\xi}), \tag{1}$$

where $\{\phi_k\}_{k=0}^{P}$ is an *orthonormal* basis of polynomial functions constructed based on the inner product

$$\langle \phi_k, \phi_m \rangle = \int_{\mathbb{R}^d} \phi_k(\boldsymbol{\xi})\phi_m(\boldsymbol{\xi})w(\boldsymbol{\xi})d\boldsymbol{\xi} = \delta_{km}, \tag{2}$$

with $w(\boldsymbol{\xi})$ the joint probability density function (PDF) of $\boldsymbol{\xi}$ and δ_{km} the Kronecker delta. For standard distributions, the classes of polynomials are well known and correspond e.g. to Hermite polynomials (for Gaussian RVs), Legendre polynomials (for uniform RVs), and so on.

The advantage of having a representation like (1) is that, thanks to the orthogonality properties, the first two statistical moments are readily given as

$$\mathrm{E}\{v(t,\boldsymbol{\xi})\} \approx v_0(t), \qquad \mathrm{Var}\{v(t,\boldsymbol{\xi})\} \approx \sum_{k=1}^{P} v_k^2(t), \tag{3}$$

and this of course also holds for the current $i(t,\boldsymbol{\xi})$. Moreover, higher order moments as well as distribution functions can be obtained by randomly sampling (1), this step being extremely fast because (1) are mere polynomials.

The problem therefore reduces to the determination of the unknown coefficients $v_k(t)$ and $i_k(t)$. It can be proven that such coefficients are the line voltages and currents of a *deterministic* multiconductor transmission line described by the following telegrapher's equations (see e.g., [10])

$$\begin{aligned} \frac{\partial}{\partial z}\tilde{\mathbf{v}}(z,t) &= -\tilde{\mathbf{R}}\tilde{\mathbf{i}}(z,t) - \tilde{\mathbf{L}}\frac{\partial}{\partial t}\tilde{\mathbf{i}}(z,t) \\ \frac{\partial}{\partial z}\tilde{\mathbf{i}}(z,t) &= -\tilde{\mathbf{G}}\tilde{\mathbf{v}}(z,t) - \tilde{\mathbf{C}}\frac{\partial}{\partial t}\tilde{\mathbf{v}}(z,t), \end{aligned} \tag{4}$$

where $\tilde{\mathbf{v}} = [v_0, \ldots, v_P]^T$ and $\tilde{\mathbf{i}} = [i_0, \ldots, i_P]^T$ and with the entries of the pertinent p.u.l. matrices given as e.g.

$$\tilde{R}_{jm} = \sum_{k=0}^{P} R_k \langle \phi_k\phi_m, \phi_j \rangle, \tag{5}$$

$j, m = 0, \ldots, P$. In (5), $\langle \phi_k\phi_m, \phi_j \rangle$ is merely a real number and the R_k are the PC-expansion coefficients of the random p.u.l. parameter, which can be computed — based on the random geometric and material properties of the line — via numerical integration techniques [10]. The remaining p.u.l. matrices $\tilde{\mathbf{L}}$, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{C}}$ are constructed analogously.

Once the expansion coefficients of the p.u.l. parameters are known, the modified matrices can be constructed and the corresponding multiconductor transmission line can be simulated e.g. in a SPICE-type circuit analysis tool to retrieve the sought-for coefficients for the voltage and current variables [10], provided that proper boundary conditions are enforced and the line terminations, as illustrated in Fig. 2 for the case $P = 2$. This step is discussed in the next section. As a result, a *single*, deterministic simulation of the modified circuit allows to collect statistical information faster compared to, e.g., running Monte Carlo sampling with a large number of simulations of the original network.

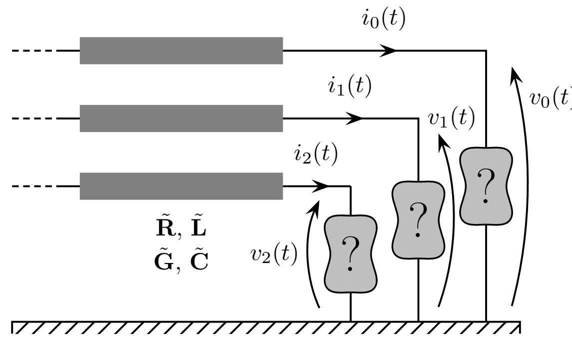

Fig. 2. Modified, deterministic transmission line and its new terminations to be determined.

III. Boundary Conditions at Line Terminations

In order to simulate the transmission line characterized by (4), suitable boundary conditions, expressed in terms of current-voltage relationships, must be derived for the line terminations. These stem from the I-V characteristic of the original termination. The result turns out to be trivial when the load is linear, because in that case such a load is simply replicated on all the terminations of the multiconductor line. Unfortunately, this is not the case when the termination is nonlinear.

Now, we relax the assumption of linearity and assume a nonlinear current-voltage relationship at the line termination:

$$i(t,\boldsymbol{\xi}) = F(v(t,\boldsymbol{\xi})). \tag{6}$$

Replacing the voltage and current with their PC expansions (1) and weighting the resulting equation using the basis functions $\{\phi_m\}$ yield

$$\sum_{k=0}^{P} i_k(t)\phi_k(\boldsymbol{\xi})\phi_m(\boldsymbol{\xi}) = F\left(\sum_{k=0}^{P} v_k(t)\phi_k(\boldsymbol{\xi})\right)\phi_m(\boldsymbol{\xi}) \tag{7}$$

($m = 0, \ldots, P$). Then, integrating with the inner product (2) produces

$$i_m(t) = \int_{\mathbb{R}^d} F\left(\sum_{k=0}^{P} v_k(t)\phi_k(\boldsymbol{\xi})\right)\phi_m(\boldsymbol{\xi})w(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{8}$$

where, in general, no exact closed-form expression exist for the right-hand side. In [11], a closed-form, but approximate, expression is obtained by discretizing the integral using a numerical quadrature with a given number of nodes. The method provides very good accuracy and excellent efficiency. However, in this paper, we provide an alternative and exact solution that applies when the nonlinear characteristic $F(v)$ can be expressed as a polynomial function, i.e.

$$i(t,\boldsymbol{\xi}) = F(v(t,\boldsymbol{\xi})) = \sum_{n=0}^{N} G_n(v(t,\boldsymbol{\xi}))^n. \tag{9}$$

Replacing (9) into (8) yields

$$i_m(t) = \int_{\mathbb{R}^d} \sum_{n=0}^{N} G_n \left(\sum_{k=0}^{P} v_k(t)\phi_k(\boldsymbol{\xi}) \right)^n \phi_m(\boldsymbol{\xi}) w(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{10}$$

The multinomial theorem allows to write

$$\left(\sum_{k=0}^{P} v_k(t)\phi_k(\boldsymbol{\xi}) \right)^n = \sum_{k_0+\ldots+k_P=n} \binom{n}{k_0,\ldots,k_P} \prod_{0\leq r\leq P} (v_r(t)\phi_r(\boldsymbol{\xi}))^{k_r}, \tag{11}$$

with the multinomial coefficient defined as

$$\binom{n}{k_0,\ldots,k_P} = \frac{n!}{k_0!\cdots k_P!}. \tag{12}$$

Substituting (11) into (10) and rearranging leads to

$$i_m(t) = \sum_{n=0}^{N} \sum_{k_0+\ldots+k_P=n} G_n \binom{n}{k_0,\ldots,k_P} \times \\ \times \prod_{0\leq r\leq P} (v_r(t))^{k_r} \int_{\mathbb{R}^d} \prod_{0\leq r\leq P} (\phi_r(\boldsymbol{\xi}))^{k_r} \phi_m(\boldsymbol{\xi}) w(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{13}$$

Despite the somewhat bulky equation, it is now possible to note that:

1) the term $\binom{n}{k_0,\ldots,k_P} G_n$ is a mere constant number;
2) the integral $\int_{\mathbb{R}^d} \prod_{0\leq r\leq P} (\phi_r(\boldsymbol{\xi}))^{k_r} \phi_m(\boldsymbol{\xi}) w(\boldsymbol{\xi}) d\boldsymbol{\xi}$ also yields a constant number that can be calculated analytically, at least for standard classes of orthogonal polynomials ϕ_k;
3) the argument $\prod_{0\leq r\leq P} (v_r(t))^{k_r}$ is still a polynomial, although $(P+1)$-variate, function of all the PC coefficients of the controlling voltage, and of total degree at most N.

As an example, the three nonlinear terminal conditions for the case $d=1$, $P=2$ and $N=2$ reduce to

$$\begin{aligned} i_0(t) &= G_0 + G_1 v_0(t) + G_2[v_0^2(t) + v_1^2(t) + v_2^2(t)] \\ i_1(t) &= G_1 v_1(t) + G_2[2v_0(t)v_1(t) + 2\sqrt{2}v_1(t)v_2(t)] \\ i_2(t) &= G_1 v_2(t) + G_2[\sqrt{2}v_1^2(t) + 2v_0(t)v_2(t) + 2\sqrt{2}v_2^2(t)] \end{aligned} \tag{14}$$

As already pointed out, the new terminal conditions (13) preserve a polynomial characteristic. This allows to take advantage of the capability, available in circuit solvers like HSPICE or PSPICE, of handling multivariate polynomial functions of a given set of controlling node voltages (using e.g. the `POLY` keyword, cfr. [12]). The new line terminations can then be straightforwardly implemented in SPICE-type simulators as voltage-dependent current sources ("G-elements") that are a polynomial function of all the line terminal voltages. It is worth noting that the statements can be parametric with respect to the coefficients G_n of the I-V characteristic, so that there is no need to redetermine the expression, should such coefficients change.

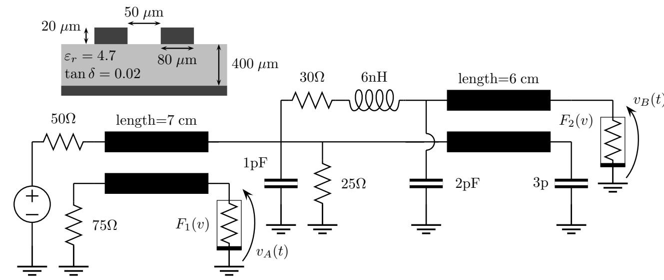

Fig. 3. Transmission-line network considered for the application example.

IV. Validation and Numerical Results

This section proposes an application example in order to better illustrate and validate the proposed theoretical formulation. The network in Fig. 3 includes two coupled copper microstrip lines with two terminations being nonlinear. The microstrip cross-section is also shown on the top. The I-V characteristics of the nonlinear terminations are $i = F_1(v) = 0.02v - 0.1v^2 + 0.6v^3$ and $i = F_2(v) = 0.01v - 0.15v^2 + 0.8v^3$. The variability is provided by the substrate thickness and by the trace-to-trace separation, which are considered as two independent Gaussian random variables with relative standard deviations of 10% and 5%, respectively. The voltage source produces a pulse with an amplitude of 5 V, a duration of 4 ns and rise/fall times of 100 ps. All the simulations are carried out using HSPICE on an ASUS U30S laptop with an Intel(R) Core(TM) i3-2330M, CPU running at 2.20 GHz and 4 GB of RAM. For additional details on how to setup a PC-based simulation of such a transmission-line network, the reader is referred to [10].

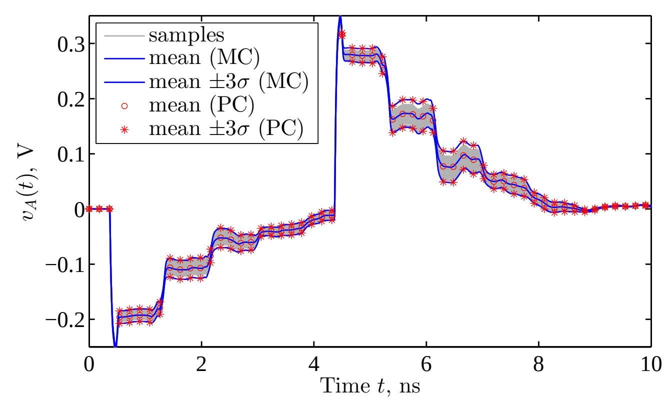

Fig. 4. Statistical transient analysis of $v_A(t)$. Gray lines: samples of the random response; blue lines: mean response and $\pm 3\sigma$ limits obtained with Monte Carlo analysis; red markers: mean response (circles) and $\pm 3\sigma$ limits (asterisks) estimated with PC.

Fig. 4 shows the transient simulation of the voltage $v_A(t)$ across the first nonlinear termination, $F_1(v)$. A 1000-run Monte Carlo analysis is performed first, using the available feature in HSPICE, and the blue lines display the corresponding estimated mean response as well as the $\pm 3\sigma$ limits (σ

denoting the standard deviation). The microstrip lines are characterized with the internal field solver, which allows to take losses and dispersion into account. A reduced set of response samples is also plotted in gray to visualize the fluctuation due to the variability of the line parameters. Then, a PC-based simulation is run to compute the PC-expansion coefficients. The red markers compare the same statistical information obtained from the voltage coefficients according to (3). A remarkable accuracy can be appreciated. Furthermore, Fig. 5 provides similar results, this time for the voltage across the second nonlinear termination, $F_2(v)$. The meaning of the curves is the same as in Fig. 4, and again excellent accuracy is established.

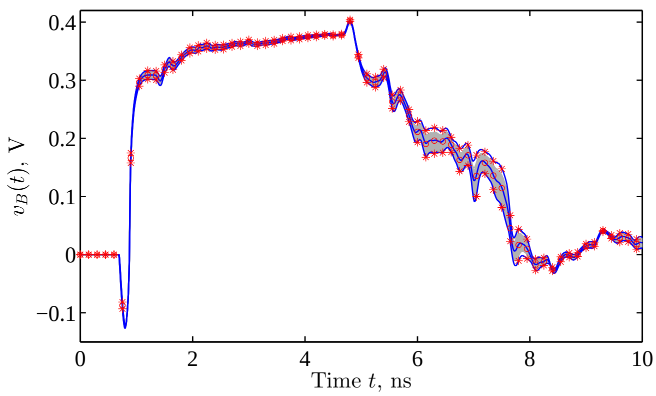

Fig. 5. Statistical transient analysis of $v_B(t)$. Curve identification is as in the inset of Fig. 4.

Finally, Fig. 6 shows the PDF of $v_A(t)$ at 6 ns, when the fluctuation of the voltage is quite large. The gray bars are the histogram constructed based on the Monte Carlo samples, whilst the red line is obtained by randomly sampling the PC expansion of $v_A(t)$. It is worthwhile to point out that this step takes less than 1 s for 10^6 samplings, nevertheless allowing a much smoother reproduction of the PDF thanks to the larger number of values considered.

As far as the simulation times are concerned, the Monte Carlo analysis required 26 min 5 s, whereas the PC simulation took 11.4 s. An impressive speed-up of 140× is thence achieved.

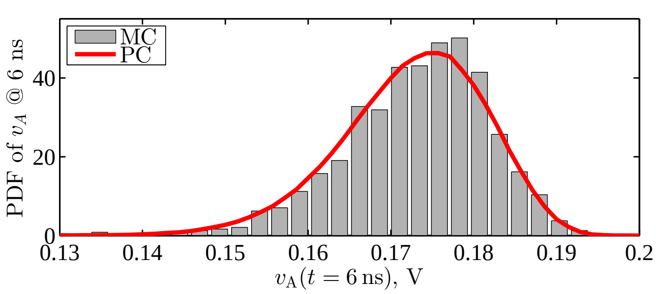

Fig. 6. Probability density function of v_A at $t = 6$ ns obtained from both the Monte Carlo samples (gray bars) and the PC expansion (red line).

V. Conclusions

This paper presents an exact and close-form formulation to include nonlinear terminations with polynomial I-V characteristic into the PC framework of statistical interconnect simulation. Contrary to the Monte Carlo approach, a single simulation of a modified network allows to extract statistical information much faster than analyzing a large number of random circuit configurations. The formulation is SPICE compatible, thus enabling the designer to perform stochastic analyses using standard circuit solvers. As such, it represents a first step towards the inclusion of nonlinear elements in the SPICE- and PC-based circuit analysis. The theory is illustrated by means of an application example involving the simulation of a network with two lossy and dispersive coupled microstrip sections having random variations in the substrate thickness and trace separation, and terminated by nonlinear loads.

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